

# Dynamical symmetry enhancement near massive IIA horizons

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## Abstract

We prove that Killing horizons in massive IIA supergravity preserve an even number of supersymmetries, and that their symmetry algebra contains an  $\mathfrak{sl}(2, \mathbb{R})$  subalgebra, confirming the conjecture of [5]. We also prove a new class of Lichnerowicz type theorems for connections of the spin bundle whose holonomy is contained in a general linear group.

# 1 Introduction

It has been known for some time that there is (super)symmetry enhancement near black hole and brane horizons. This has been observed on a case by case basis, see e.g. [1, 2, 3], and it has been extensively used in the development of the AdS/CFT correspondence [4]. Recently, it has been realized that the (super)symmetry enhancement near Killing horizons is a generic phenomenon which depends only on the smoothness of the fields and some global assumptions on the spatial horizon sections. The concise conjecture has been stated in [5], following some earlier results in [6] and [7]. This conjecture includes all the (super)symmetry enhancement phenomena near black hole Killing horizons as special cases. So far the conjecture has been verified in a variety of theories which include the minimal 5-dimensional gauged supergravity, M-theory, and IIB and IIA supergravities [6, 5, 7, 8].

In this paper, we shall prove the conjecture of [5] for the massive IIA horizons, i.e. the Killing horizons of massive IIA supergravity. This in particular implies that massive IIA horizons with smooth fields and spatial horizon sections,  $\mathcal{S}$ , which are compact without boundary:

- Preserve an even number of supersymmetries

$$N = 2N_- , \tag{1.1}$$

where  $N_-$  is the dimension of the kernel of a Dirac like operator  $\mathcal{D}^{(-)}$  on  $\mathcal{S}$  which depends on the fluxes.

- The symmetry group of all such horizons contains an  $\mathfrak{sl}(2, \mathbb{R})$  subalgebra.

The proof of the conjecture for massive IIA horizons is similar to that given in [8] for standard IIA horizons but there is a key difference. Massive IIA supergravity has a negative cosmological constant. The proof of the conjecture relies on the application of the maximum principle to demonstrate certain Lichnerowicz type theorems. In turn the application of the maximum principle requires the positive semi-definiteness of a certain term which depends the fluxes. The existence of a negative cosmological constant in the theory has the potential of invalidating the arguments based on the maximum principle as it can contribute with the opposite sign in the expressions required for the application of the maximum principle. We show that this is not the case and therefore the conjecture can be extended to massive IIA horizons.

Nevertheless many of the steps in the proof of the conjecture for massive IIA horizons are similar to those presented for IIA horizons in [8]. Because of this, in the main body of the paper, we shall state the key statements and formulae required for the proof of the conjecture. The detailed proofs of these are presented in the appendices.

This paper is organised as follows. In section 2, we show that massive IIA horizons preserve an even number of supersymmetries. In section 3, we demonstrate that the symmetry of massive IIA horizons includes an  $\mathfrak{sl}(2, \mathbb{R})$  subalgebra. In addition, in appendix A, we give the field equations of the near horizon fields. In appendix B, we identify the independent KSEs of the near horizon geometries. In appendix C, we derive some key

formulae which are required for the proof of Lichnerowicz type theorems for  $\mathcal{D}^{(\pm)}$  operators, and in appendix D we present some identities which are necessary to demonstrate the  $\mathfrak{sl}(2, \mathbb{R})$  invariance of massive IIA horizons.

## 2 Supersymmetry enhancement

### 2.1 Independent KSEs

The first part of the conjecture states that massive IIA horizons preserve an even number of supersymmetries. In particular, if the massive IIA horizons admit one supersymmetry, then this enhances to two. To prove this, we solve the KSEs of massive IIA supergravity [11]

$$\begin{aligned} \mathcal{D}_\mu \epsilon \equiv & \nabla_\mu \epsilon + \frac{1}{8} H_{\mu\nu_1\nu_2} \Gamma^{\nu_1\nu_2} \Gamma_{11} \epsilon + \frac{1}{16} e^\Phi F_{\nu_1\nu_2} \Gamma^{\nu_1\nu_2} \Gamma_\mu \Gamma_{11} \epsilon \\ & + \frac{1}{8 \cdot 4!} e^\Phi G_{\nu_1\nu_2\nu_3\nu_4} \Gamma^{\nu_1\nu_2\nu_3\nu_4} \Gamma_\mu \epsilon + \frac{1}{8} e^\Phi m \Gamma_\mu \epsilon = 0 , \end{aligned} \quad (2.1)$$

$$\begin{aligned} \mathcal{A} \epsilon \equiv & \partial_\mu \Phi \Gamma^\mu \epsilon + \frac{1}{12} H_{\mu_1\mu_2\mu_3} \Gamma^{\mu_1\mu_2\mu_3} \Gamma_{11} \epsilon + \frac{3}{8} e^\Phi F_{\mu_1\mu_2} \Gamma^{\mu_1\mu_2} \Gamma_{11} \epsilon \\ & + \frac{1}{4 \cdot 4!} e^\Phi G_{\mu_1\mu_2\mu_3\mu_4} \Gamma^{\mu_1\mu_2\mu_3\mu_4} \epsilon + \frac{5}{4} e^\Phi m \epsilon = 0 , \end{aligned} \quad (2.2)$$

for the near horizon fields

$$\begin{aligned} ds^2 &= 2\mathbf{e}^+ \mathbf{e}^- + \delta_{ij} \mathbf{e}^i \mathbf{e}^j , \quad G = \mathbf{e}^+ \wedge \mathbf{e}^- \wedge X + r \mathbf{e}^+ \wedge Y + \tilde{G} , \\ H &= \mathbf{e}^+ \wedge \mathbf{e}^- \wedge L + r \mathbf{e}^+ \wedge M + \tilde{H} , \quad F = \mathbf{e}^+ \wedge \mathbf{e}^- S + r \mathbf{e}^+ \wedge T + \tilde{F} , \end{aligned} \quad (2.3)$$

where  $\epsilon$  is a commuting Majorana  $Spin(9, 1)$  spinor and we have introduced the frame

$$\mathbf{e}^+ = du, \quad \mathbf{e}^- = dr + rh - \frac{1}{2} r^2 \Delta du, \quad \mathbf{e}^i = e_I^i dy^I . \quad (2.4)$$

This expression for the near horizon fields is similar to that for the IIA case in [8] though their dependence on the gauge potentials is different. The massive theory contains an additional parameter  $m$ , the mass term, and the fields and both the gravitino and dilatino KSEs depend on it, see appendix A. Furthermore, the Bianchi identities relate some of the components of the near horizon fields. In particular,  $M$ ,  $T$  and  $Y$  are not independent, see again appendix A. The dependence on the coordinates  $u, r$  is given explicitly and all the fields depend on the coordinates  $y^I$  of the spatial horizon section  $\mathcal{S}$  defined by  $u = r = 0$ .

The KSEs of massive IIA supergravity can be solved along the lightcone directions. The solution is

$$\epsilon = \epsilon_+ + \epsilon_- , \quad \epsilon_+ = \phi_+(u, y) , \quad \epsilon_- = \phi_- + r \Gamma_- \Theta_+ \phi_+ , \quad (2.5)$$

and

$$\phi_- = \eta_- , \quad \phi_+ = \eta_+ + u \Gamma_+ \Theta_- \eta_- , \quad (2.6)$$

where

$$\begin{aligned}\Theta_{\pm} &= \frac{1}{4}h_i\Gamma^i \mp \frac{1}{4}\Gamma_{11}L_i\Gamma^i - \frac{1}{16}e^{\Phi}\Gamma_{11}(\pm 2S + \tilde{F}_{ij}\Gamma^{ij}) \\ &- \frac{1}{8 \cdot 4!}e^{\Phi}(\pm 12X_{ij}\Gamma^{ij} + \tilde{G}_{ijkl}\Gamma^{ijkl}) - \frac{1}{8}e^{\Phi}m,\end{aligned}\tag{2.7}$$

$\Gamma_{\pm}\epsilon_{\pm} = 0$ , and  $\eta_{\pm} = \eta_{\pm}(y)$  depend only on the coordinates  $y$  of the spatial horizon section  $\mathcal{S}$ . Both  $\eta_{\pm}$  are sections of the  $Spin(8)$  bundle over  $\mathcal{S}$  associated with the Majorana representation.

Substituting the spinor  $\epsilon$  given in (2.5) into the KSEs (2.1) and (2.2), one obtains a large number of conditions given in appendix B. To describe the remaining independent KSEs consider the operators

$$\nabla_i^{(\pm)} = \tilde{\nabla}_i + \Psi_i^{(\pm)},\tag{2.8}$$

with

$$\begin{aligned}\Psi_i^{(\pm)} &= \left( \mp \frac{1}{4}h_i \mp \frac{1}{16}e^{\Phi}X_{l_1l_2}\Gamma^{l_1l_2}\Gamma_i + \frac{1}{8 \cdot 4!}e^{\Phi}\tilde{G}_{l_1l_2l_3l_4}\Gamma^{l_1l_2l_3l_4}\Gamma_i + \frac{1}{8}e^{\Phi}m\Gamma_i \right) \\ &+ \Gamma_{11}\left( \mp \frac{1}{4}L_i + \frac{1}{8}\tilde{H}_{il_1l_2}\Gamma^{l_1l_2} \pm \frac{1}{8}e^{\Phi}S\Gamma_i - \frac{1}{16}e^{\Phi}\tilde{F}_{l_1l_2}\Gamma^{l_1l_2}\Gamma_i \right),\end{aligned}\tag{2.9}$$

and

$$\begin{aligned}\mathcal{A}^{(\pm)} &= \partial_i\Phi\Gamma^i + \left( \mp \frac{1}{8}e^{\Phi}X_{l_1l_2}\Gamma^{l_1l_2} + \frac{1}{4 \cdot 4!}e^{\Phi}\tilde{G}_{l_1l_2l_3l_4}\Gamma^{l_1l_2l_3l_4} + \frac{5}{4}e^{\Phi}m \right) \\ &+ \Gamma_{11}\left( \pm \frac{1}{2}L_i\Gamma^i - \frac{1}{12}\tilde{H}_{ijk}\Gamma^{ijk} \mp \frac{3}{4}e^{\Phi}S + \frac{3}{8}e^{\Phi}\tilde{F}_{ij}\Gamma^{ij} \right).\end{aligned}\tag{2.10}$$

These are derived from the naive restriction of the supercovariant derivative and the dilatino KSE on  $\mathcal{S}$ .

*Theorem:* The remaining independent KSEs are

$$\nabla_i^{(\pm)}\eta_{\pm} = 0, \quad \mathcal{A}^{(\pm)}\eta_{\pm} = 0.\tag{2.11}$$

Moreover if  $\eta_-$  solves the KSEs, then

$$\eta_+ = \Gamma_+\Theta_-\eta_-, \tag{2.12}$$

is also a solution.

*Proof:* The proof is given in appendix B.

□

## 2.2 Lichnerowicz type theorems for $\mathcal{D}^{(\pm)}$

To proceed with the proof of the first part of the conjecture define the modified horizon Dirac operators as

$$\mathcal{D}^{(\pm)} = \mathcal{D}^{(\pm)} - \mathcal{A}^{(\pm)} , \quad (2.13)$$

where

$$\mathcal{D}^{(\pm)} \equiv \Gamma^i \nabla_i^{(\pm)} = \Gamma^i \tilde{\nabla}_i + \Psi^{(\pm)} , \quad (2.14)$$

with

$$\begin{aligned} \Psi^{(\pm)} \equiv \Gamma^i \Psi_i^{(\pm)} &= \mp \frac{1}{4} h_i \Gamma^i \mp \frac{1}{4} e^\Phi X_{ij} \Gamma^{ij} + e^\Phi m \\ &+ \Gamma_{11} \left( \pm \frac{1}{4} L_i \Gamma^i - \frac{1}{8} \tilde{H}_{ijk} \Gamma^{ijk} \mp e^\Phi S + \frac{1}{4} e^\Phi \tilde{F}_{ij} \Gamma^{ij} \right) , \end{aligned} \quad (2.15)$$

are the horizon Dirac operators associated with the supercovariant derivatives  $\nabla^{(\pm)}$ .

*Theorem:* Let  $\mathcal{S}$  and the fields satisfy the conditions for the maximum principle to apply, e.g. the fields are smooth and  $\mathcal{S}$  is compact without boundary. Then there is a 1-1 correspondence between the zero modes of  $\mathcal{D}^{(+)}$  and the  $\eta_+$  Killing spinors, i.e.

$$\nabla_i^{(+)} \eta_+ = 0 , \quad \mathcal{A}^{(+)} \eta_+ = 0 \iff \mathcal{D}^{(+)} \eta_+ = 0 . \quad (2.16)$$

Moreover  $\|\eta_+\|^2$  is constant.

*Proof:* It is evident that if  $\eta_+$  is a Killing spinor, then it is a zero mode of  $\mathcal{D}^{(+)}$ . To prove the converse, assuming that  $\eta_+$  is a zero mode of  $\mathcal{D}^{(+)}$  and after using the field equations and Bianchi identities, one can establish the identity, see appendix C,

$$\tilde{\nabla}^i \tilde{\nabla}_i \|\eta_+\|^2 - (2\tilde{\nabla}^i \Phi + h^i) \tilde{\nabla}_i \|\eta_+\|^2 = 2 \|\hat{\nabla}^{(+)} \eta_+\|^2 + (-4\kappa - 16\kappa^2) \|\mathcal{A}^{(+)} \eta_+\|^2 \quad (2.17)$$

where

$$\hat{\nabla}_i^{(\pm)} = \nabla_i^{(\pm)} + \kappa \Gamma_i \mathcal{A}^{(\pm)} , \quad (2.18)$$

for some  $\kappa \in \mathbb{R}$ . Provided that  $\kappa$  is chosen in the interval  $(-\frac{1}{4}, 0)$ , the theorem follows as an application of the maximum principle.  $\square$

Let us turn to investigate the relation between Killing spinors and the zero modes of the  $\mathcal{D}^{(-)}$  operator.

*Theorem:* Let  $\mathcal{S}$  be compact without boundary and the horizon fields be smooth. There is a 1-1 correspondence between the zero modes of  $\mathcal{D}^{(-)}$  and the  $\eta_-$  Killing spinors, i.e.

$$\nabla_i^{(-)} \eta_- = 0 , \quad \mathcal{A}^{(-)} \eta_- = 0 \iff \mathcal{D}^{(-)} \eta_- = 0 . \quad (2.19)$$

*Proof:* It is clear that if  $\eta_-$  is a Killing spinor, then it is a zero mode of  $\mathcal{D}^{(-)}$ . To prove the converse, if  $\eta_-$  is a zero mode of  $\mathcal{D}^{(-)}$ , then upon using the field equations and Bianchi identities one can establish the formula, see appendix C,

$$\tilde{\nabla}^i(e^{-2\Phi}V_i) = -2e^{-2\Phi} \|\hat{\nabla}^{(-)}\eta_-\|^2 + e^{-2\Phi}(4\kappa + 16\kappa^2) \|\mathcal{A}^{(-)}\eta_-\|^2, \quad (2.20)$$

where  $V = -d \|\eta_-\|^2 - \|\eta_-\|^2 h$ . The theorem follows after integrating the above formula over  $\mathcal{S}$  using Stokes' theorem for  $\kappa \in (-\frac{1}{4}, 0)$ .  $\square$

## 2.3 Index theory and supersymmetry enhancement

To prove the first part of the conjecture, we shall establish the theorem:

*Theorem:* The number of supersymmetries preserved by massive IIA horizons is even.

*Proof:* Let  $N_{\pm}$  be the number of  $\eta_{\pm}$  Killing spinors. As a consequence of the two theorems we have established in the previous section  $N_{\pm} = \dim \text{Ker } \mathcal{D}^{(\pm)}$ . The  $Spin(9, 1)$  bundle over the spacetime decomposes as  $S_+ \oplus S_-$  upon restriction to  $\mathcal{S}$ . Furthermore  $S_+$  and  $S_-$  are isomorphic as  $Spin(8)$  bundles as both are associated with the Majorana representation. The action of  $\mathcal{D}^{(+)} : \Gamma(S_+) \rightarrow \Gamma(S_+)$  on the section  $\Gamma(S_+)$  of  $S_+$  is not chirality preserving. Since the principal symbol of  $\mathcal{D}^{(+)}$  is the same as the principal symbol of the standard Dirac operator acting on Majorana but not-Weyl spinors, the index vanishes [13]. Therefore

$$N_+ = \dim \text{Ker } \mathcal{D}^{(+)} = \dim \text{Ker } (\mathcal{D}^{(+)})^{\dagger}, \quad (2.21)$$

where  $(\mathcal{D}^{(+)})^{\dagger}$  is the adjoint of  $\mathcal{D}^{(+)}$ . On the other hand, one can establish

$$(e^{2\Phi}\Gamma_-)(\mathcal{D}^{(+)})^{\dagger} = \mathcal{D}^{(-)}(e^{2\Phi}\Gamma_-), \quad (2.22)$$

and so

$$N_- = \dim \text{Ker } (\mathcal{D}^{(-)}) = \dim \text{Ker } (\mathcal{D}^{(+)})^{\dagger}. \quad (2.23)$$

Therefore, we conclude that  $N_+ = N_-$  and so the number of supersymmetries of massive IIA horizons  $N = N_+ + N_- = 2N_-$  is even.  $\square$

## 3 The $\mathfrak{sl}(2, \mathbb{R})$ symmetry of massive IIA horizons

### 3.1 $\eta_+$ from $\eta_-$ Killing spinors

We shall demonstrate the existence of the  $\mathfrak{sl}(2, \mathbb{R})$  symmetry of massive IIA horizons by directly constructing the vector fields on the spacetime generated by the action of  $\mathfrak{sl}(2, \mathbb{R})$ . In turn the existence of such vector fields is a consequence of the property that massive

IIA horizons admit an even number of supersymmetries. We have seen that if  $\eta_-$  is a Killing spinor, then  $\eta_+ = \Gamma_+ \Theta_- \eta_-$  is also a Killing spinor provided that  $\eta_+ \neq 0$ . It turns out that under certain conditions this is always possible.

*Lemma:* Suppose that  $\mathcal{S}$  and the fields satisfy the requirements for the maximum principle to apply. Then

$$\text{Ker } \Theta_- = \{0\} . \quad (3.1)$$

*Proof:* We shall prove this by contradiction. Assume that  $\Theta_-$  has a non-trivial kernel, so there is  $\eta_- \neq 0$  such that  $\Theta_- \eta_- = 0$ . In such a case, (B.3) gives  $\Delta \langle \eta_-, \eta_- \rangle = 0$ . Thus  $\Delta = 0$ , as  $\eta_-$  is no-where vanishing.

Next the gravitino KSE  $\nabla^{(-)} \eta_- = 0$  together with  $\langle \eta_-, \Gamma_i \Theta_- \eta_- \rangle = 0$  imply that

$$\tilde{\nabla}_i \|\eta_-\|^2 = -h_i \|\eta_-\|^2 . \quad (3.2)$$

On taking the divergence of this expression, eliminating  $\tilde{\nabla}^i h_i$  upon using (A.17), and after setting  $\Delta = 0$ , one finds

$$\begin{aligned} \tilde{\nabla}^i \tilde{\nabla}_i \|\eta_-\|^2 &= 2\tilde{\nabla}^i \Phi \tilde{\nabla}_i \|\eta_-\|^2 \\ &+ \left( L^2 + \frac{1}{2} e^{2\Phi} S^2 + \frac{1}{4} e^{2\Phi} X^2 + \frac{1}{4} e^{2\Phi} \tilde{F}^2 + \frac{1}{48} e^{2\Phi} \tilde{G}^2 + \frac{1}{2} e^{2\Phi} m^2 \right) \|\eta_-\|^2 . \end{aligned} \quad (3.3)$$

The maximum principle implies that  $\|\eta_-\|^2$  is constant. However, the remainder of (3.3) can never vanish, due to the quadratic term in  $m$ . So there can be no solutions, with  $m \neq 0$ , such that  $\eta_- \neq 0$  is in the Kernel of  $\Theta_-$ , and so  $\text{Ker } \Theta_- = \{0\}$ .  $\square$

### 3.2 $\mathfrak{sl}(2, \mathbb{R})$ symmetry

Using  $\eta_-$  and  $\eta_+ = \Gamma_+ \Theta_- \eta_-$  and the formula (2.5), one can construct two linearly independent Killing spinors on the spacetime as

$$\epsilon_1 = \eta_- + u\eta_+ + ru\Gamma_- \Theta_+ \eta_+ , \quad \epsilon_2 = \eta_+ + r\Gamma_- \Theta_+ \eta_+ . \quad (3.4)$$

It is known from the general theory of supersymmetric massive IIA backgrounds that for any Killing spinors  $\zeta_1$  and  $\zeta_2$  the dual vector field  $K(\zeta_1, \zeta_2)$  of the 1-form bilinear

$$\omega(\zeta_1, \zeta_2) = \langle (\Gamma_+ - \Gamma_-) \zeta_1, \Gamma_a \zeta_2 \rangle e^a , \quad (3.5)$$

is a Killing vector and leaves invariant all the other fields of the theory. Evaluating the vector field bilinears of the Killing spinors  $\epsilon_1$  and  $\epsilon_2$ , we find that

$$\begin{aligned} K_1(\epsilon_1, \epsilon_2) &= -2u \|\eta_+\|^2 \partial_u + 2r \|\eta_+\|^2 \partial_r + \tilde{V} , \\ K_2(\epsilon_2, \epsilon_2) &= -2 \|\eta_+\|^2 \partial_u , \\ K_3(\epsilon_1, \epsilon_1) &= -2u^2 \|\eta_+\|^2 \partial_u + (2 \|\eta_-\|^2 + 4ru \|\eta_+\|^2) \partial_r + 2u\tilde{V} , \end{aligned} \quad (3.6)$$

where we have set

$$\tilde{V} = \langle \Gamma_+ \eta_-, \Gamma^i \eta_+ \rangle \tilde{\partial}_i , \quad (3.7)$$

is a vector field on  $\mathcal{S}$ . To derive the above expressions for the Killing vector fields, we have used the identities

$$-\Delta \|\eta_+\|^2 + 4 \|\Theta_+ \eta_+\|^2 = 0 , \quad \langle \eta_+, \Gamma_i \Theta_+ \eta_+ \rangle = 0 , \quad (3.8)$$

which follow from the first integrability condition in (B.1),  $\|\eta_+\| = \text{const}$  and the KSEs of  $\eta_+$ .

*Theorem:* The Lie bracket algebra of  $K_1$ ,  $K_2$  and  $K_3$  is  $\mathfrak{sl}(2, \mathbb{R})$ .

*Proof:* Using the identities summarised in appendix D, one can demonstrate after a direct computation that

$$[K_1, K_2] = 2 \|\eta_+\|^2 K_2 , \quad [K_2, K_3] = -4 \|\eta_+\|^2 K_1 , \quad [K_3, K_1] = 2 \|\eta_+\|^2 K_3 . \quad (3.9)$$

This proves the theorem and the last part of the conjecture. □

## Acknowledgements

UG is supported by the Knut and Alice Wallenberg Foundation. GP is partially supported by the STFC grant ST/J002798/1. JG is supported by the STFC grant, ST/1004874/1. JG would like to thank the Department of Mathematical Sciences, University of Liverpool for hospitality during which part of this work was completed. UK is supported by a STFC PhD fellowship.

## Appendix A Horizon Field equations and Bianchi Identities

The bosonic fields of massive IIA supergravity [11] are the spacetime metric  $g$ , the dilaton  $\Phi$ , the 2-form NS-NS gauge potential  $B$ , and the 1-form and the 3-form RR gauge potentials  $A$  and  $C$ , respectively. The theory also includes a mass parameter  $m$  which induces a negative cosmological constant in the theory. In addition, fermionic fields of the theory are a Majorana gravitino and dilatino which are set to zero in all the computations that follow. The bosonic field strengths of massive IIA supergravity [11] in the conventions of [12] are

$$F = dA + mB , \quad H = dB , \quad G = dC - H \wedge A + \frac{1}{2} mB \wedge B , \quad (A.1)$$

implying the Bianchi identities

$$dF = mH , \quad dH = 0 , \quad dG = F \wedge H . \quad (A.2)$$



The bosonic part of the massive IIA action in the string frame is

$$\begin{aligned}
S = \int \Big[ & d^{10}x \sqrt{-g} \Big( e^{-2\Phi} \Big( R + 4\nabla_\mu \Phi \nabla^\mu \Phi - \frac{1}{12} H_{\lambda_1 \lambda_2 \lambda_3} H^{\lambda_1 \lambda_2 \lambda_3} \Big) \\
& - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{48} G_{\mu_1 \mu_2 \mu_3 \mu_4} G^{\mu_1 \mu_2 \mu_3 \mu_4} - \frac{1}{2} m^2 \Big) \\
& + \frac{1}{2} dC \wedge dC \wedge B + \frac{m}{6} dC \wedge B \wedge B \wedge B \\
& + \frac{m^2}{40} B \wedge B \wedge B \wedge B \wedge B \Big] . \tag{A.3}
\end{aligned}$$

This leads to the Einstein equation

$$\begin{aligned}
R_{\mu\nu} = & -2\nabla_\mu \nabla_\nu \Phi + \frac{1}{4} H_{\mu\lambda_1\lambda_2} H_{\nu}{}^{\lambda_1\lambda_2} + \frac{1}{2} e^{2\Phi} F_{\mu\lambda} F_{\nu}{}^{\lambda} + \frac{1}{12} e^{2\Phi} G_{\mu\lambda_1\lambda_2\lambda_3} G_{\nu}{}^{\lambda_1\lambda_2\lambda_3} \\
& + g_{\mu\nu} \left( -\frac{1}{8} e^{2\Phi} F_{\lambda_1\lambda_2} F^{\lambda_1\lambda_2} - \frac{1}{96} e^{2\Phi} G_{\lambda_1\lambda_2\lambda_3\lambda_4} G^{\lambda_1\lambda_2\lambda_3\lambda_4} - \frac{1}{4} e^{2\Phi} m^2 \right) , \tag{A.4}
\end{aligned}$$

and the dilaton field equation

$$\begin{aligned}
\nabla^\mu \nabla_\mu \Phi = & 2\nabla_\lambda \Phi \nabla^\lambda \Phi - \frac{1}{12} H_{\lambda_1\lambda_2\lambda_3} H^{\lambda_1\lambda_2\lambda_3} + \frac{3}{8} e^{2\Phi} F_{\lambda_1\lambda_2} F^{\lambda_1\lambda_2} \\
& + \frac{1}{96} e^{2\Phi} G_{\lambda_1\lambda_2\lambda_3\lambda_4} G^{\lambda_1\lambda_2\lambda_3\lambda_4} + \frac{5}{4} e^{2\Phi} m^2 , \tag{A.5}
\end{aligned}$$

the 2-form field equation

$$\nabla^\mu F_{\mu\nu} + \frac{1}{6} H^{\lambda_1\lambda_2\lambda_3} G_{\lambda_1\lambda_2\lambda_3\nu} = 0 , \tag{A.6}$$

the 3-form field equation

$$\nabla_\lambda \left( e^{-2\Phi} H^{\lambda\mu\nu} \right) - m F^{\mu\nu} - \frac{1}{2} G^{\mu\nu\lambda_1\lambda_2} F_{\lambda_1\lambda_2} + \frac{1}{1152} \epsilon^{\mu\nu\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6\lambda_7\lambda_8} G_{\lambda_1\lambda_2\lambda_3\lambda_4} G_{\lambda_5\lambda_6\lambda_7\lambda_8} = 0 , \tag{A.7}$$

and the 4-form field equation

$$\nabla_\mu G^{\mu\nu_1\nu_2\nu_3} + \frac{1}{144} \epsilon^{\nu_1\nu_2\nu_3\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6\lambda_7} G_{\lambda_1\lambda_2\lambda_3\lambda_4} H_{\lambda_5\lambda_6\lambda_7} = 0 . \tag{A.8}$$

Adapting Gaussian null coordinates [9, 10] near massive IIA Killing horizons, one finds

$$\begin{aligned}
ds^2 = & 2\mathbf{e}^+ \mathbf{e}^- + \delta_{ij} \mathbf{e}^i \mathbf{e}^j , \quad G = \mathbf{e}^+ \wedge \mathbf{e}^- \wedge X + r \mathbf{e}^+ \wedge Y + \tilde{G} , \\
H = & \mathbf{e}^+ \wedge \mathbf{e}^- \wedge L + r \mathbf{e}^+ \wedge M + \tilde{H} , \quad F = \mathbf{e}^+ \wedge \mathbf{e}^- S + r \mathbf{e}^+ \wedge T + \tilde{F} , \tag{A.9}
\end{aligned}$$

where  $\Delta$  is a function,  $h$ ,  $L$  and  $T$  are 1-forms,  $X$ ,  $M$  and  $\tilde{F}$  are 2-forms,  $Y$ ,  $\tilde{H}$  are 3-forms and  $\tilde{G}$  is a 4-form on the spatial horizon section  $\mathcal{S}$ . The dilaton  $\Phi$  is also taken as a function on  $\mathcal{S}$ .

Substituting the fields (2.3) into the Bianchi identities of massive IIA supergravity, one finds that

$$\begin{aligned} M &= d_h L, \quad T = d_h S - m L, \quad Y = d_h X - L \wedge \tilde{F} - S \tilde{H}, \\ d\tilde{G} &= \tilde{H} \wedge \tilde{F}, \quad d\tilde{H} = 0, \quad d\tilde{F} = m\tilde{H}, \end{aligned} \quad (\text{A.10})$$

where  $d_h \theta \equiv d\theta - h \wedge \theta$  for any form  $\theta$ .

Similarly, the independent field equations of the near horizon fields are as follows. The 2-form field equation (A.6) gives

$$\tilde{\nabla}^i \tilde{F}_{ik} - h^i \tilde{F}_{ik} + T_k - L^i X_{ik} + \frac{1}{6} \tilde{H}^{\ell_1 \ell_2 \ell_3} \tilde{G}_{\ell_1 \ell_2 \ell_3 k} = 0, \quad (\text{A.11})$$

the 3-form field equation (A.7) gives

$$\tilde{\nabla}^i (e^{-2\Phi} L_i) - m S - \frac{1}{2} \tilde{F}^{ij} X_{ij} + \frac{1}{1152} \epsilon^{\ell_1 \ell_2 \ell_3 \ell_4 \ell_5 \ell_6 \ell_7 \ell_8} \tilde{G}_{\ell_1 \ell_2 \ell_3 \ell_4} \tilde{G}_{\ell_5 \ell_6 \ell_7 \ell_8} = 0, \quad (\text{A.12})$$

and

$$\begin{aligned} \tilde{\nabla}^i (e^{-2\Phi} \tilde{H}_{imn}) - m \tilde{F}_{mn} - e^{-2\Phi} h^i \tilde{H}_{imn} + e^{-2\Phi} M_{mn} + S X_{mn} - \frac{1}{2} \tilde{F}^{ij} \tilde{G}_{ijmn} \\ - \frac{1}{48} \epsilon_{mn}^{\ell_1 \ell_2 \ell_3 \ell_4 \ell_5 \ell_6} X_{\ell_1 \ell_2} \tilde{G}_{\ell_3 \ell_4 \ell_5 \ell_6} = 0, \end{aligned} \quad (\text{A.13})$$

and the 4-form field equation (A.8) gives

$$\tilde{\nabla}^i X_{ik} + \frac{1}{144} \epsilon_k^{\ell_1 \ell_2 \ell_3 \ell_4 \ell_5 \ell_6 \ell_7} \tilde{G}_{\ell_1 \ell_2 \ell_3 \ell_4} \tilde{H}_{\ell_5 \ell_6 \ell_7} = 0, \quad (\text{A.14})$$

and

$$\tilde{\nabla}^i \tilde{G}_{ijkq} + Y_{jkq} - h^i \tilde{G}_{ijkq} - \frac{1}{12} \epsilon_{jkq}^{\ell_1 \ell_2 \ell_3 \ell_4 \ell_5} X_{\ell_1 \ell_2} \tilde{H}_{\ell_3 \ell_4 \ell_5} - \frac{1}{24} \epsilon_{jkq}^{\ell_1 \ell_2 \ell_3 \ell_4 \ell_5} \tilde{G}_{\ell_1 \ell_2 \ell_3 \ell_4} L_{\ell_5} = 0, \quad (\text{A.15})$$

where  $\tilde{\nabla}$  is the Levi-Civita connection of the metric on  $\mathcal{S}$ . In addition, the dilaton field equation (A.5) becomes

$$\begin{aligned} \tilde{\nabla}^i \tilde{\nabla}_i \Phi - h^i \tilde{\nabla}_i \Phi &= 2 \tilde{\nabla}_i \Phi \tilde{\nabla}^i \Phi + \frac{1}{2} L_i L^i - \frac{1}{12} \tilde{H}_{\ell_1 \ell_2 \ell_3} \tilde{H}^{\ell_1 \ell_2 \ell_3} - \frac{3}{4} e^{2\Phi} S^2 \\ &+ \frac{3}{8} e^{2\Phi} \tilde{F}_{ij} \tilde{F}^{ij} - \frac{1}{8} e^{2\Phi} X_{ij} X^{ij} + \frac{1}{96} e^{2\Phi} \tilde{G}_{\ell_1 \ell_2 \ell_3 \ell_4} \tilde{G}^{\ell_1 \ell_2 \ell_3 \ell_4} + \frac{5}{4} e^{2\Phi} m^2. \end{aligned} \quad (\text{A.16})$$

It remains to evaluate the Einstein field equation. This gives

$$\begin{aligned} \frac{1}{2} \tilde{\nabla}^i h_i - \Delta - \frac{1}{2} h^2 &= h^i \tilde{\nabla}_i \Phi - \frac{1}{2} L_i L^i - \frac{1}{4} e^{2\Phi} S^2 - \frac{1}{8} e^{2\Phi} X_{ij} X^{ij} \\ &- \frac{1}{8} e^{2\Phi} \tilde{F}_{ij} \tilde{F}^{ij} - \frac{1}{96} e^{2\Phi} \tilde{G}_{\ell_1 \ell_2 \ell_3 \ell_4} \tilde{G}^{\ell_1 \ell_2 \ell_3 \ell_4} - \frac{1}{4} e^{2\Phi} m^2, \end{aligned} \quad (\text{A.17})$$

and

$$\begin{aligned}
\tilde{R}_{ij} = & -\tilde{\nabla}_{(i}h_{j)} + \frac{1}{2}h_i h_j - 2\tilde{\nabla}_i \tilde{\nabla}_j \Phi - \frac{1}{2}L_i L_j + \frac{1}{4}\tilde{H}_{i\ell_1\ell_2}\tilde{H}_j{}^{\ell_1\ell_2} \\
& + \frac{1}{2}e^{2\Phi}\tilde{F}_{i\ell}\tilde{F}_j{}^\ell - \frac{1}{2}e^{2\Phi}X_{i\ell}X_j{}^\ell + \frac{1}{12}e^{2\Phi}\tilde{G}_{i\ell_1\ell_2\ell_3}\tilde{G}_j{}^{\ell_1\ell_2\ell_3} \\
& + \delta_{ij}\left(\frac{1}{4}e^{2\Phi}S^2 - \frac{1}{4}e^{2\Phi}m^2 - \frac{1}{8}e^{2\Phi}\tilde{F}_{\ell_1\ell_2}\tilde{F}^{\ell_1\ell_2} + \frac{1}{8}e^{2\Phi}X_{\ell_1\ell_2}X^{\ell_1\ell_2} - \frac{1}{96}e^{2\Phi}\tilde{G}_{\ell_1\ell_2\ell_3\ell_4}\tilde{G}^{\ell_1\ell_2\ell_3\ell_4}\right),
\end{aligned} \tag{A.18}$$

where  $\tilde{R}$  denotes the Ricci tensor of  $\mathcal{S}$ .

There are additional Bianchi identities and field equations which however are not independent of those we have stated above. We give these because they are useful in many of the intermediate computations. In particular, we have the additional Bianchi identities

$$\begin{aligned}
dT + Sdh + dS \wedge h + mdL &= 0, \\
dM + L \wedge dh - h \wedge dL &= 0, \\
dY + dh \wedge X - h \wedge dX + h \wedge (S\tilde{H} + \tilde{F} \wedge L) + T \wedge \tilde{H} + \tilde{F} \wedge M &= 0.
\end{aligned} \tag{A.19}$$

There are also additional field equations given by

$$-\tilde{\nabla}^i T_i + h^i T_i - \frac{1}{2}dh^{ij}\tilde{F}_{ij} - \frac{1}{2}X_{ij}M^{ij} - \frac{1}{6}Y_{ijk}\tilde{H}^{ijk} = 0, \tag{A.20}$$

$$\begin{aligned}
-\tilde{\nabla}^i(e^{-2\Phi}M_{ik}) + e^{-2\Phi}h^i M_{ik} - \frac{1}{2}e^{-2\Phi}dh^{ij}\tilde{H}_{ijk} - T^i X_{ik} - \frac{1}{2}\tilde{F}^{ij}Y_{ijk} \\
-mT_k - \frac{1}{144}\epsilon_k{}^{\ell_1\ell_2\ell_3\ell_4\ell_5\ell_6\ell_7}Y_{\ell_1\ell_2\ell_3}\tilde{G}_{\ell_4\ell_5\ell_6\ell_7} = 0,
\end{aligned} \tag{A.21}$$

$$\begin{aligned}
-\tilde{\nabla}^i Y_{imn} + h^i Y_{imn} - \frac{1}{2}dh^{ij}\tilde{G}_{ijmn} + \frac{1}{36}\epsilon_{mn}{}^{\ell_1\ell_2\ell_3\ell_4\ell_5\ell_6}Y_{\ell_1\ell_2\ell_3}\tilde{H}_{\ell_4\ell_5\ell_6} \\
+\frac{1}{48}\epsilon_{mn}{}^{\ell_1\ell_2\ell_3\ell_4\ell_5\ell_6}\tilde{G}_{\ell_1\ell_2\ell_3\ell_4}M_{\ell_5\ell_6} = 0,
\end{aligned} \tag{A.22}$$

corresponding to equations obtained from the  $+$  component of (A.6), the  $k$  component of (A.7) and the  $mn$  component of (A.8) respectively. However, (A.20), (A.21) and (A.22) are implied by (A.11)-(A.15) together with the Bianchi identities (A.10).

Note also that the  $++$  and  $+i$  components of the Einstein equation, which are

$$\begin{aligned}
\frac{1}{2}\tilde{\nabla}^i \tilde{\nabla}_i \Delta - \frac{3}{2}h^i \tilde{\nabla}_i \Delta - \frac{1}{2}\Delta \tilde{\nabla}^i h_i + \Delta h^2 + \frac{1}{4}dh_{ij}dh^{ij} &= (\tilde{\nabla}^i \Delta - \Delta h^i)\tilde{\nabla}_i \Phi + \frac{1}{4}M_{ij}M^{ij} \\
&+ \frac{1}{2}e^{2\Phi}T_i T^i + \frac{1}{12}e^{2\Phi}Y_{ijk}Y^{ijk}
\end{aligned} \tag{A.23}$$

and

$$\begin{aligned} \frac{1}{2}\tilde{\nabla}^j dh_{ij} - dh_{ij}h^j - \tilde{\nabla}_i\Delta + \Delta h_i &= dh_i{}^j\tilde{\nabla}_j\Phi - \frac{1}{2}M_i{}^jL_j + \frac{1}{4}M_{\ell_1\ell_2}\tilde{H}_i{}^{\ell_1\ell_2} - \frac{1}{2}e^{2\Phi}ST_i \\ &+ \frac{1}{2}e^{2\Phi}T^j\tilde{F}_{ij} - \frac{1}{4}e^{2\Phi}Y_i{}^{\ell_1\ell_2}X_{\ell_1\ell_2} + \frac{1}{12}e^{2\Phi}Y_{\ell_1\ell_2\ell_3}\tilde{G}_i{}^{\ell_1\ell_2\ell_3} , \end{aligned} \quad (\text{A.24})$$

are implied by (A.16), (A.17), (A.18), together with (A.11)-(A.15), and the Bianchi identities (A.10).

## B Integrability conditions and KSEs

Substituting the solution (2.5) of the KSEs along the light cone directions back into the gravitino KSE (2.1), and appropriately expanding in the  $r$  and  $u$  coordinates, we find that for the  $\mu = \pm$  components, one obtains the additional conditions

$$\begin{aligned} &\left( \frac{1}{2}\Delta - \frac{1}{8}(dh)_{ij}\Gamma^{ij} + \frac{1}{8}M_{ij}\Gamma_{11}\Gamma^{ij} + 2\left(\frac{1}{4}h_i\Gamma^i - \frac{1}{4}L_i\Gamma_{11}\Gamma^i \right. \right. \\ &\left. \left. - \frac{1}{16}e^\Phi\Gamma_{11}(-2S + \tilde{F}_{ij}\Gamma^{ij}) - \frac{1}{8 \cdot 4!}e^\Phi(12X_{ij}\Gamma^{ij} - \tilde{G}_{ijkl}\Gamma^{ijkl}) + \frac{1}{8}e^\Phi m)\Theta_+ \right) \phi_+ = 0 , \end{aligned} \quad (\text{B.1})$$

$$\left( \frac{1}{4}\Delta h_i\Gamma^i - \frac{1}{4}\partial_i\Delta\Gamma^i + \left( -\frac{1}{8}(dh)_{ij}\Gamma^{ij} - \frac{1}{8}M_{ij}\Gamma^{ij}\Gamma_{11} - \frac{1}{4}e^\Phi T_i\Gamma^i\Gamma_{11} + \frac{1}{24}e^\Phi Y_{ijk}\Gamma^{ijk} \right) \Theta_+ \right) \phi_+ = 0 , \quad (\text{B.2})$$

$$\begin{aligned} &\left( -\frac{1}{2}\Delta - \frac{1}{8}(dh)_{ij}\Gamma^{ij} + \frac{1}{8}M_{ij}\Gamma^{ij}\Gamma_{11} - \frac{1}{4}e^\Phi T_i\Gamma^i\Gamma_{11} - \frac{1}{24}e^\Phi Y_{ijk}\Gamma^{ijk} + 2\left( -\frac{1}{4}h_i\Gamma^i \right. \right. \\ &\left. \left. - \frac{1}{4}\Gamma_{11}L_i\Gamma^i + \frac{1}{16}e^\Phi\Gamma_{11}(2S + \tilde{F}_{ij}\Gamma^{ij}) - \frac{1}{8 \cdot 4!}e^\Phi(12X_{ij}\Gamma^{ij} + \tilde{G}_{ijkl}\Gamma^{ijkl}) - \frac{1}{8}e^\Phi m)\Theta_- \right) \phi_- = 0 . \end{aligned} \quad (\text{B.3})$$

Similarly the  $\mu = i$  component of the gravitino KSEs gives

$$\nabla_i^{(\pm)}\phi_\pm = 0 \quad (\text{B.4})$$

and

$$\begin{aligned} &\tilde{\nabla}_i\tau_+ + \left( -\frac{3}{4}h_i - \frac{1}{16}e^\Phi X_{l_1l_2}\Gamma^{l_1l_2}\Gamma_i - \frac{1}{8 \cdot 4!}e^\Phi \tilde{G}_{l_1\dots l_4}\Gamma^{l_1\dots l_4}\Gamma_i - \frac{1}{8}e^\Phi m\Gamma_i \right. \\ &\left. - \Gamma_{11}\left(\frac{1}{4}L_i + \frac{1}{8}\tilde{H}_{ijk}\Gamma^{jk} + \frac{1}{8}e^\Phi S\Gamma_i + \frac{1}{16}e^\Phi \tilde{F}_{l_1l_2}\Gamma^{l_1l_2}\Gamma_i\right) \tau_+ \right. \\ &\left. + \left( -\frac{1}{4}(dh)_{ij}\Gamma^j - \frac{1}{4}M_{ij}\Gamma^j\Gamma_{11} + \frac{1}{8}e^\Phi T_j\Gamma^j\Gamma_i\Gamma_{11} + \frac{1}{48}e^\Phi Y_{l_1l_2l_3}\Gamma^{l_1l_2l_3}\Gamma_i \right) \phi_+ = 0 , \end{aligned} \quad (\text{B.5})$$

where we have set

$$\tau_+ = \Theta_+ \phi_+ . \quad (\text{B.6})$$

We shall demonstrate that all the above conditions are not independent and follow upon using the field equations and the Bianchi identities from those in (2.11).

Similarly, substituting the solution of the KSEs (2.5) into the dilatino KSE (2.2) and expanding appropriately in the  $r$  and  $u$  coordinates, we find

$$\begin{aligned} & \partial_i \Phi \Gamma^i \phi_\pm - \frac{1}{12} \Gamma_{11} (\mp 6 L_i \Gamma^i + \tilde{H}_{ijk} \Gamma^{ijk}) \phi_\pm + \frac{3}{8} e^\Phi \Gamma_{11} (\mp 2 S + \tilde{F}_{ij} \Gamma^{ij}) \phi_\pm \\ & + \frac{1}{4 \cdot 4!} e^\Phi (\mp 12 X_{ij} \Gamma^{ij} + \tilde{G}_{j_1 j_2 j_3 j_4} \Gamma^{j_1 j_2 j_3 j_4}) \phi_\pm + \frac{5}{4} e^\Phi m \phi_\pm = 0 , \end{aligned} \quad (\text{B.7})$$

$$\begin{aligned} & - \left( \partial_i \Phi \Gamma^i + \frac{1}{12} \Gamma_{11} (6 L_i \Gamma^i + \tilde{H}_{ijk} \Gamma^{ijk}) + \frac{3}{8} e^\Phi \Gamma_{11} (2 S + \tilde{F}_{ij} \Gamma^{ij}) \right. \\ & \left. - \frac{1}{4 \cdot 4!} e^\Phi (12 X_{ij} \Gamma^{ij} + \tilde{G}_{ijkl} \Gamma^{ijkl}) - \frac{5}{4} e^\Phi m \right) \tau_+ \\ & + \left( \frac{1}{4} M_{ij} \Gamma^{ij} \Gamma_{11} + \frac{3}{4} e^\Phi T_i \Gamma^i \Gamma_{11} + \frac{1}{24} e^\Phi Y_{ijk} \Gamma^{ijk} \right) \phi_+ = 0 . \end{aligned} \quad (\text{B.8})$$

Again, these are not independent of those in (2.11).

## B.1 Independent KSEs

### B.1.1 The (B.5) condition

The (B.5) component of the KSEs is implied by (B.4), (B.6) and (B.7) together with a number of field equations and Bianchi identities. First evaluate the LHS of (B.5) by substituting in (B.6) to eliminate  $\tau_+$ , and use (B.4) to evaluate the supercovariant derivative of  $\phi_+$ . Also, using (B.4) one can compute

$$\begin{aligned} & (\tilde{\nabla}_j \tilde{\nabla}_i - \tilde{\nabla}_i \tilde{\nabla}_j) \phi_+ = \frac{1}{4} \tilde{\nabla}_j (h_i) \phi_+ + \frac{1}{4} \Gamma_{11} \tilde{\nabla}_j (L_i) \phi_+ - \frac{1}{8} \Gamma_{11} \tilde{\nabla}_j (\tilde{H}_{il_1 l_2}) \Gamma^{l_1 l_2} \phi_+ \\ & + \frac{1}{16} e^\Phi \Gamma_{11} (-2 \tilde{\nabla}_j (S) + \tilde{\nabla}_j (\tilde{F}_{kl}) \Gamma^{kl}) \Gamma_i \phi_+ - \frac{1}{8 \cdot 4!} e^\Phi (-12 \tilde{\nabla}_j (X_{kl}) \Gamma^{kl} + \tilde{\nabla}_j (\tilde{G}_{j_1 j_2 j_3 j_4}) \Gamma^{j_1 j_2 j_3 j_4}) \Gamma_i \phi_+ \\ & + \frac{1}{16} \tilde{\nabla}_j \Phi e^\Phi \Gamma_{11} (-2 S + \tilde{F}_{kl} \Gamma^{kl}) \Gamma_i \phi_+ - \frac{1}{8 \cdot 4!} \tilde{\nabla}_j \Phi e^\Phi (-12 X_{kl} \Gamma^{kl} + \tilde{G}_{j_1 j_2 j_3 j_4} \Gamma^{j_1 j_2 j_3 j_4}) \Gamma_i \phi_+ \\ & - \frac{1}{8} e^\Phi \tilde{\nabla}_j \Phi m \Gamma_i \phi_+ + \left( \frac{1}{4} h_i + \frac{1}{4} \Gamma_{11} L_i - \frac{1}{8} \Gamma_{11} \tilde{H}_{ijk} \Gamma^{jk} + \frac{1}{16} e^\Phi \Gamma_{11} (-2 S + \tilde{F}_{kl} \Gamma^{kl}) \Gamma_i \right. \\ & \left. - \frac{1}{8 \cdot 4!} e^\Phi (-12 X_{kl} \Gamma^{kl} + \tilde{G}_{j_1 j_2 j_3 j_4} \Gamma^{j_1 j_2 j_3 j_4}) \Gamma_i - \frac{1}{8} e^\Phi m \Gamma_i \right) \tilde{\nabla}_j \phi_+ - (i \leftrightarrow j) . \end{aligned} \quad (\text{B.9})$$

Then consider the following, where the first terms cancel from the definition of curvature,

$$\left( \frac{1}{4} \tilde{R}_{ij} \Gamma^j - \frac{1}{2} \Gamma^j (\tilde{\nabla}_j \tilde{\nabla}_i - \tilde{\nabla}_i \tilde{\nabla}_j) \right) \phi_+ + \frac{1}{2} \tilde{\nabla}_i (\mathcal{A}_1) + \frac{1}{2} \Psi_i \mathcal{A}_1 = 0 , \quad (\text{B.10})$$

where

$$\mathcal{A}_1 = \partial_i \Phi \Gamma^i \phi_+ - \frac{1}{12} \Gamma_{11} (-6 L_i \Gamma^i + \tilde{H}_{ijk} \Gamma^{ijk}) \phi_+ + \frac{3}{8} e^\Phi \Gamma_{11} (-2 S + \tilde{F}_{ij} \Gamma^{ij}) \phi_+$$

$$+ \frac{1}{4 \cdot 4!} e^\Phi (-12 X_{ij} \Gamma^{ij} + \tilde{G}_{j_1 j_2 j_3 j_4} \Gamma^{j_1 j_2 j_3 j_4}) \phi_+ + \frac{5}{4} e^\Phi m \phi_+ \quad (\text{B.11})$$

and

$$\Psi_i = -\frac{1}{4} h_i + \Gamma_{11} \left( \frac{1}{4} L_i - \frac{1}{8} \tilde{H}_{ijk} \Gamma^{jk} \right). \quad (\text{B.12})$$

The expression in (B.11) vanishes on making use of (B.7), as  $\mathcal{A}_1 = 0$  is equivalent to the  $+$  component of (B.7). However a non-trivial identity is obtained by using (B.9) in (B.10), and expanding out the  $\mathcal{A}_1$  terms. Then, on adding (B.10) to the LHS of (B.5), with  $\tau_+$  eliminated in favour of  $\eta_+$  as described above, one obtains the following

$$\begin{aligned} & \frac{1}{4} \left( \tilde{R}_{ij} + \tilde{\nabla}_{(i} h_{j)} - \frac{1}{2} h_i h_j + 2 \tilde{\nabla}_i \tilde{\nabla}_j \Phi + \frac{1}{2} L_i L_j - \frac{1}{4} \tilde{H}_{il_1 l_2} \tilde{H}_j{}^{l_1 l_2} \right. \\ & - \frac{1}{2} e^{2\Phi} \tilde{F}_{il} \tilde{F}_j{}^l + \frac{1}{8} e^{2\Phi} \tilde{F}_{l_1 l_2} \tilde{F}{}^{l_1 l_2} \delta_{ij} + \frac{1}{2} e^{2\Phi} X_{il} X_j{}^l - \frac{1}{8} e^{2\Phi} X_{l_1 l_2} X{}^{l_1 l_2} \delta_{ij} \\ & \left. - \frac{1}{12} e^{2\Phi} \tilde{G}_{il_1 l_2 l_3} \tilde{G}_j{}^{l_1 l_2 l_3} + \frac{1}{96} e^{2\Phi} \tilde{G}_{l_1 l_2 l_3 l_4} \tilde{G}{}^{l_1 l_2 l_3 l_4} \delta_{ij} - \frac{1}{4} e^{2\Phi} S^2 \delta_{ij} + \frac{1}{4} e^{2\Phi} m^2 \delta_{ij} \right) \Gamma^j = 0. \end{aligned} \quad (\text{B.13})$$

This vanishes identically on making use of the Einstein equation (A.18). Therefore it follows that (B.5) is implied by the  $+$  component of (B.4), (B.6) and (B.7), the Bianchi identities (A.10) and the gauge field equations (A.11)-(A.15).

### B.1.2 The (B.8) condition

Let us define

$$\begin{aligned} \mathcal{A}_2 = & - \left( \partial_i \Phi \Gamma^i + \frac{1}{12} \Gamma_{11} (6 L_i \Gamma^i + \tilde{H}_{ijk} \Gamma^{ijk}) + \frac{3}{8} e^\Phi \Gamma_{11} (2S + \tilde{F}_{ij} \Gamma^{ij}) \right. \\ & - \frac{1}{4 \cdot 4!} e^\Phi (12 X_{ij} \Gamma^{ij} + \tilde{G}_{ijkl} \Gamma^{ijkl}) - \frac{5}{4} e^\Phi m \Big) \tau_+ \\ & + \left( \frac{1}{4} M_{ij} \Gamma^{ij} \Gamma_{11} + \frac{3}{4} e^\Phi T_i \Gamma^i \Gamma_{11} + \frac{1}{24} e^\Phi Y_{ijk} \Gamma^{ijk} \right) \phi_+, \end{aligned} \quad (\text{B.14})$$

where  $\mathcal{A}_2$  equals the expression in (B.8). One obtains the following identity

$$\mathcal{A}_2 = -\frac{1}{2} \Gamma^i \tilde{\nabla}_i \mathcal{A}_1 + \Psi_1 \mathcal{A}_1, \quad (\text{B.15})$$

where

$$\begin{aligned} \Psi_1 = & \tilde{\nabla}_i \Phi \Gamma^i + \frac{3}{8} h_i \Gamma^i + \frac{1}{16} e^\Phi X_{l_1 l_2} \Gamma^{l_1 l_2} - \frac{1}{192} e^\Phi \tilde{G}_{l_1 l_2 l_3 l_4} \Gamma^{l_1 l_2 l_3 l_4} - \frac{1}{8} e^\Phi m \\ & + \Gamma_{11} \left( \frac{1}{48} \tilde{H}_{l_1 l_2 l_3} \Gamma^{l_1 l_2 l_3} - \frac{1}{8} L_i \Gamma^i + \frac{1}{16} e^\Phi \tilde{F}_{l_1 l_2} \Gamma^{l_1 l_2} - \frac{1}{8} e^\Phi S \right). \end{aligned} \quad (\text{B.16})$$

We have made use of the  $+$  component of (B.4) in order to evaluate the covariant derivative in the above expression. In addition we have made use of the Bianchi identities (A.10) and the field equations (A.11)-(A.16).

### B.1.3 The (B.1) condition

In order to show that (B.1) is implied by the independent KSEs we can compute the following,

$$\begin{aligned} & \left( -\frac{1}{4}\tilde{R} - \Gamma^{ij}\tilde{\nabla}_i\tilde{\nabla}_j \right) \phi_+ - \Gamma^i\tilde{\nabla}_i(\mathcal{A}_1) \\ & + \left( \tilde{\nabla}_i\Phi\Gamma^i + \frac{1}{4}h_i\Gamma^i + \frac{1}{16}e^\Phi X_{l_1l_2}\Gamma^{l_1l_2} - \frac{1}{192}e^\Phi\tilde{G}_{l_1l_2l_3l_4}\Gamma^{l_1l_2l_3l_4} - \frac{1}{8}e^\Phi m \right. \\ & \left. + \Gamma_{11}\left(-\frac{1}{4}L_l\Gamma^l - \frac{1}{24}\tilde{H}_{l_1l_2l_3}\Gamma^{l_1l_2l_3} - \frac{1}{8}e^\Phi S + \frac{1}{16}e^\Phi\tilde{F}_{l_1l_2}\Gamma^{l_1l_2}\right) \right) \mathcal{A}_1 = 0, \end{aligned} \quad (\text{B.17})$$

where

$$\begin{aligned} \tilde{R} &= -2\Delta - 2h^i\tilde{\nabla}_i\Phi - 2\tilde{\nabla}^2\Phi - \frac{1}{2}h^2 + \frac{1}{2}L^2 + \frac{1}{4}\tilde{H}^2 + \frac{5}{2}e^{2\Phi}S^2 \\ &\quad - \frac{1}{4}e^{2\Phi}\tilde{F}^2 + \frac{3}{4}e^{2\Phi}X^2 + \frac{1}{48}e^{2\Phi}\tilde{G}^2 - \frac{3}{2}e^{2\Phi}m^2 \end{aligned} \quad (\text{B.18})$$

and where we use the + component of (B.4) to evaluate the covariant derivative terms. In order to obtain (B.1) from these expressions we make use of the Bianchi identities (A.10), the field equations (A.11)-(A.16), in particular in order to eliminate the  $(\tilde{\nabla}\Phi)^2$  term. We have also made use of the +- component of the Einstein equation (A.17) in order to rewrite the scalar curvature  $\tilde{R}$  in terms of  $\Delta$ . Therefore (B.1) follows from (B.4) and (B.7) together with the field equations and Bianchi identities mentioned above.

### B.1.4 The + (B.7) condition linear in $u$

Since  $\phi_+ = \eta_+ + u\Gamma_+\Theta_-\eta_-$ , we must consider the part of the + component of (B.7) which is linear in  $u$ . On defining

$$\begin{aligned} \mathcal{B}_1 &= \partial_i\Phi\Gamma^i\eta_- - \frac{1}{12}\Gamma_{11}(6L_i\Gamma^i + \tilde{H}_{ijk}\Gamma^{ijk})\eta_- + \frac{3}{8}e^\Phi\Gamma_{11}(2S + \tilde{F}_{ij}\Gamma^{ij})\eta_- \\ &\quad + \frac{1}{4\cdot 4!}e^\Phi(12X_{ij}\Gamma^{ij} + \tilde{G}_{j_1j_2j_3j_4}\Gamma^{j_1j_2j_3j_4})\eta_- + \frac{5}{4}e^\Phi m \eta_- \end{aligned} \quad (\text{B.19})$$

one finds that the  $u$ -dependent part of (B.7) is proportional to

$$-\frac{1}{2}\Gamma^i\tilde{\nabla}_i(\mathcal{B}_1) + \Psi_2\mathcal{B}_1, \quad (\text{B.20})$$

where

$$\begin{aligned} \Psi_2 &= \tilde{\nabla}_i\Phi\Gamma^i + \frac{1}{8}h_i\Gamma^i - \frac{1}{16}e^\Phi X_{l_1l_2}\Gamma^{l_1l_2} - \frac{1}{192}e^\Phi\tilde{G}_{l_1l_2l_3l_4}\Gamma^{l_1l_2l_3l_4} - \frac{1}{8}e^\Phi m \\ &\quad + \Gamma_{11}\left(\frac{1}{48}\tilde{H}_{l_1l_2l_3}\Gamma^{l_1l_2l_3} + \frac{1}{8}L_i\Gamma^i + \frac{1}{16}e^\Phi\tilde{F}_{l_1l_2}\Gamma^{l_1l_2} + \frac{1}{8}e^\Phi S\right). \end{aligned} \quad (\text{B.21})$$

We have made use of the - component of (B.4) in order to evaluate the covariant derivative in the above expression. In addition we have made use of the Bianchi identities (A.10) and the field equations (A.11)-(A.16).

### B.1.5 The (B.2) condition

In order to show that (B.2) is implied by the independent KSEs we will show that it follows from (B.1). First act on (B.1) with the Dirac operator  $\Gamma^i \tilde{\nabla}_i$  and use the field equations (A.11) - (A.16) and the Bianchi identities to eliminate the terms which contain derivatives of the fluxes and then use (B.1) to rewrite the  $dh$ -terms in terms of  $\Delta$ . Then use the conditions (B.4) and (B.5) to eliminate the  $\partial_i \Phi$ -terms from the resulting expression, some of the remaining terms will vanish as a consequence of (B.1). After performing these calculations, the condition (B.2) is obtained, therefore it follows from section B.1.3 above that (B.2) is implied by (B.4) and (B.7) together with the field equations and Bianchi identities mentioned above.

### B.1.6 The (B.3) condition

In order to show that (B.3) is implied by the independent KSEs we can compute the following,

$$\begin{aligned} & \left( \frac{1}{4} \tilde{R} + \Gamma^{ij} \tilde{\nabla}_i \tilde{\nabla}_j \right) \eta_- + \Gamma^i \tilde{\nabla}_i (\mathcal{B}_1) \\ & + \left( -\tilde{\nabla}_i \Phi \Gamma^i + \frac{1}{4} h_i \Gamma^i + \frac{1}{16} e^\Phi X_{l_1 l_2} \Gamma^{l_1 l_2} + \frac{1}{192} e^\Phi \tilde{G}_{l_1 l_2 l_3 l_4} \Gamma^{l_1 l_2 l_3 l_4} + \frac{1}{8} e^\Phi m \right. \\ & \left. + \Gamma_{11} \left( -\frac{1}{4} L_l \Gamma^l + \frac{1}{24} \tilde{H}_{l_1 l_2 l_3} \Gamma^{l_1 l_2 l_3} - \frac{1}{8} e^\Phi S - \frac{1}{16} e^\Phi \tilde{F}_{l_1 l_2} \Gamma^{l_1 l_2} \right) \right) \mathcal{B}_1 = 0, \end{aligned} \quad (\text{B.22})$$

where we use the  $-$  component of (B.4) to evaluate the covariant derivative terms. The expression above vanishes identically since the  $-$  component of (B.7) is equivalent to  $\mathcal{B}_1 = 0$ . In order to obtain (B.3) from these expressions we make use of the Bianchi identities (A.10) and the field equations (A.11)-(A.16). Therefore (B.3) follows from (B.4) and (B.7) together with the field equations and Bianchi identities mentioned above.

### B.1.7 The $+$ (B.4) condition linear in $u$

Next consider the part of the  $+$  component of (B.4) which is linear in  $u$ . First compute

$$\left( \Gamma^j (\tilde{\nabla}_j \tilde{\nabla}_i - \tilde{\nabla}_i \tilde{\nabla}_j) - \frac{1}{2} \tilde{R}_{ij} \Gamma^j \right) \eta_- - \tilde{\nabla}_i (\mathcal{B}_1) - \Psi_i \mathcal{B}_1 = 0, \quad (\text{B.23})$$

where

$$\Psi_i = \frac{1}{4} h_i - \Gamma_{11} \left( \frac{1}{4} L_i + \frac{1}{8} \tilde{H}_{ijk} \Gamma^{jk} \right) \quad (\text{B.24})$$

and where we have made use of the  $-$  component of (B.4) to evaluate the covariant derivative terms. The resulting expression corresponds to the expression obtained by expanding out the  $u$ -dependent part of the  $+$  component of (B.4) by using the  $-$  component of (B.4) to evaluate the covariant derivative. We have made use of the Bianchi identities (A.10) and the field equations (A.11)-(A.15).



## Appendix C Calculation of Laplacian of $\| \eta_{\pm} \|^2$

To establish the Lichnerowicz type theorems in 2.2, we calculate the Laplacian of  $\| \eta_{\pm} \|^2$ . For this let us generalise the modified horizon Dirac operator as  $\mathcal{D}^{(\pm)} = \mathcal{D}^{(\pm)} + q\mathcal{A}^{(\pm)}$  and assume throughout that  $\mathcal{D}^{(\pm)}\eta_{\pm} = 0$ ; in section 2.2 we had set  $q = -1$ .

To proceed, we compute the Laplacian

$$\tilde{\nabla}^i \tilde{\nabla}_i \| \eta_{\pm} \|^2 = 2\langle \eta_{\pm}, \tilde{\nabla}^i \tilde{\nabla}_i \eta_{\pm} \rangle + 2\langle \tilde{\nabla}^i \eta_{\pm}, \tilde{\nabla}_i \eta_{\pm} \rangle . \quad (\text{C.1})$$

To evaluate this expression note that

$$\begin{aligned} \tilde{\nabla}^i \tilde{\nabla}_i \eta_{\pm} &= \Gamma^i \tilde{\nabla}_i (\Gamma^j \tilde{\nabla}_j \eta_{\pm}) - \Gamma^{ij} \tilde{\nabla}_i \tilde{\nabla}_j \eta_{\pm} \\ &= \Gamma^i \tilde{\nabla}_i (\Gamma^j \tilde{\nabla}_j \eta_{\pm}) + \frac{1}{4} \tilde{R} \eta_{\pm} \\ &= \Gamma^i \tilde{\nabla}_i (-\Psi^{(\pm)} \eta_{\pm} - q\mathcal{A}^{(\pm)} \eta_{\pm}) + \frac{1}{4} \tilde{R} \eta_{\pm} . \end{aligned} \quad (\text{C.2})$$

It follows that

$$\begin{aligned} \langle \eta_{\pm}, \tilde{\nabla}^i \tilde{\nabla}_i \eta_{\pm} \rangle &= \frac{1}{4} \tilde{R} \| \eta_{\pm} \|^2 + \langle \eta_{\pm}, \Gamma^i \tilde{\nabla}_i (-\Psi^{(\pm)} - q\mathcal{A}^{(\pm)}) \eta_{\pm} \rangle \\ &+ \langle \eta_{\pm}, \Gamma^i (-\Psi^{(\pm)} - q\mathcal{A}^{(\pm)}) \tilde{\nabla}_i \eta_{\pm} \rangle , \end{aligned} \quad (\text{C.3})$$

and also

$$\begin{aligned} \langle \tilde{\nabla}^i \eta_{\pm}, \tilde{\nabla}_i \eta_{\pm} \rangle &= \langle \hat{\nabla}^{(\pm)i} \eta_{\pm}, \hat{\nabla}_i^{(\pm)} \eta_{\pm} \rangle - 2\langle \eta_{\pm}, (\Psi^{(\pm)i} + \kappa \Gamma^i \mathcal{A}^{(\pm)})^{\dagger} \tilde{\nabla}_i \eta_{\pm} \rangle \\ &- \langle \eta_{\pm}, (\Psi^{(\pm)i} + \kappa \Gamma^i \mathcal{A}^{(\pm)})^{\dagger} (\Psi_i^{(\pm)} + \kappa \Gamma_i \mathcal{A}^{(\pm)}) \eta_{\pm} \rangle \\ &= \| \hat{\nabla}^{(\pm)} \eta_{\pm} \|^2 - 2\langle \eta_{\pm}, \Psi^{(\pm)i\dagger} \tilde{\nabla}_i \eta_{\pm} \rangle - 2\kappa \langle \eta_{\pm}, \mathcal{A}^{(\pm)\dagger} \Gamma^i \tilde{\nabla}_i \eta_{\pm} \rangle \\ &- \langle \eta_{\pm}, (\Psi^{(\pm)i\dagger} \Psi_i^{(\pm)} + 2\kappa \mathcal{A}^{(\pm)\dagger} \Psi^{(\pm)} + 8\kappa^2 \mathcal{A}^{(\pm)\dagger} \mathcal{A}^{(\pm)}) \eta_{\pm} \rangle \\ &= \| \hat{\nabla}^{(\pm)} \eta_{\pm} \|^2 - 2\langle \eta_{\pm}, \Psi^{(\pm)i\dagger} \tilde{\nabla}_i \eta_{\pm} \rangle - \langle \eta_{\pm}, \Psi^{(\pm)i\dagger} \Psi_i^{(\pm)} \eta_{\pm} \rangle \\ &+ (2\kappa q - 8\kappa^2) \| \mathcal{A}^{(\pm)} \eta_{\pm} \|^2 . \end{aligned} \quad (\text{C.4})$$

Therefore,

$$\begin{aligned} \frac{1}{2} \tilde{\nabla}^i \tilde{\nabla}_i \| \eta_{\pm} \|^2 &= \| \hat{\nabla}^{(\pm)} \eta_{\pm} \|^2 + (2\kappa q - 8\kappa^2) \| \mathcal{A}^{(\pm)} \eta_{\pm} \|^2 \\ &+ \langle \eta_{\pm}, \left( \frac{1}{4} \tilde{R} + \Gamma^i \tilde{\nabla}_i (-\Psi^{(\pm)} - q\mathcal{A}^{(\pm)}) - \Psi^{(\pm)i\dagger} \Psi_i^{(\pm)} \right) \eta_{\pm} \rangle \\ &+ \langle \eta_{\pm}, \left( \Gamma^i (-\Psi^{(\pm)} - q\mathcal{A}^{(\pm)}) - 2\Psi^{(\pm)i\dagger} \right) \tilde{\nabla}_i \eta_{\pm} \rangle . \end{aligned} \quad (\text{C.5})$$

In order to simplify the expression for the Laplacian, we shall attempt to rewrite the third line in (C.5) as

$$\langle \eta_{\pm}, \left( \Gamma^i (-\Psi^{(\pm)} - q\mathcal{A}^{(\pm)}) - 2\Psi^{(\pm)i\dagger} \right) \tilde{\nabla}_i \eta_{\pm} \rangle = \langle \eta_{\pm}, \mathcal{F}^{(\pm)} \Gamma^i \tilde{\nabla}_i \eta_{\pm} \rangle + W^{(\pm)i} \tilde{\nabla}_i \| \eta_{\pm} \|^2 , \quad (\text{C.6})$$

where  $\mathcal{F}^{(\pm)}$  is linear in the fields and  $W^{(\pm)i}$  is a vector. This expression is particularly advantageous, because the first term on the RHS can be rewritten using the horizon Dirac equation, and the second term is consistent with the application of the maximum principle/integration by parts arguments which are required for the generalised Lichnerowicz theorems. In order to rewrite (C.6) in this fashion, note that

$$\begin{aligned}
\Gamma^i(\Psi^{(\pm)} + q\mathcal{A}^{(\pm)}) + 2\Psi^{(\pm)i\dagger} &= (\mp h^i \mp (q+1)\Gamma_{11}L^i + \frac{1}{2}(q+1)\Gamma_{11}\tilde{H}^i_{\ell_1\ell_2}\Gamma^{\ell_1\ell_2} + 2q\tilde{\nabla}^i\Phi) \\
&+ (\pm \frac{1}{4}h_j\Gamma^j \pm (\frac{q}{2} + \frac{1}{4})\Gamma_{11}L_j\Gamma^j \\
&- (\frac{q}{12} + \frac{1}{8})\Gamma_{11}\tilde{H}_{\ell_1\ell_2\ell_3}\Gamma^{\ell_1\ell_2\ell_3} - q\tilde{\nabla}_j\Phi\Gamma^j)\Gamma^i \\
&+ (q+1)\left(\mp \frac{1}{8}e^\Phi X_{\ell_1\ell_2}\Gamma^i\Gamma^{\ell_1\ell_2} + \frac{1}{96}e^\Phi\tilde{G}_{\ell_1\ell_2\ell_3\ell_4}\Gamma^i\Gamma^{\ell_1\ell_2\ell_3\ell_4} + \frac{5}{4}e^\Phi m\Gamma^i\right) \\
&+ (q+1)\Gamma_{11}\left(\pm \frac{3}{4}e^\Phi S\Gamma^i - \frac{3}{8}e^\Phi\tilde{F}_{\ell_1\ell_2}\Gamma^i\Gamma^{\ell_1\ell_2}\right). \tag{C.7}
\end{aligned}$$

One finds that (C.6) is only possible for  $q = -1$  and thus we have

$$W^{(\pm)i} = \frac{1}{2}(2\tilde{\nabla}^i\Phi \pm h^i) \tag{C.8}$$

$$\mathcal{F}^{(\pm)} = \mp \frac{1}{4}h_j\Gamma^j - \tilde{\nabla}_j\Phi\Gamma^j + \Gamma_{11}\left(\pm \frac{1}{4}L_j\Gamma^j + \frac{1}{24}\tilde{H}_{\ell_1\ell_2\ell_3}\Gamma^{\ell_1\ell_2\ell_3}\right). \tag{C.9}$$

We remark that  $\dagger$  is the adjoint with respect to the  $Spin(8)$ -invariant inner product  $\langle , \rangle$ . The choice of inner product is such that

$$\begin{aligned}
\langle \eta_+, \Gamma^{[k]}\eta_+ \rangle &= 0, & k = 2 \pmod{4} \text{ and } k = 3 \pmod{4} \\
\langle \eta_+, \Gamma_{11}\Gamma^{[k]}\eta_+ \rangle &= 0, & k = 1 \pmod{4} \text{ and } k = 2 \pmod{4},
\end{aligned} \tag{C.10}$$

where  $\Gamma^{[k]}$  denote skew-symmetric products of  $k$  gamma matrices. For a more detailed explanation see [8].

It follows that

$$\begin{aligned}
\frac{1}{2}\tilde{\nabla}^i\tilde{\nabla}_i\|\eta_\pm\|^2 &= \|\hat{\nabla}^{(\pm)}\eta_\pm\|^2 + (-2\kappa - 8\kappa^2)\|\mathcal{A}^{(\pm)}\eta_\pm\|^2 + W^{(\pm)i}\tilde{\nabla}_i\|\eta_\pm\|^2 \\
&+ \langle \eta_\pm, \left(\frac{1}{4}\tilde{R} + \Gamma^i\tilde{\nabla}_i(-\Psi^{(\pm)} + \mathcal{A}^{(\pm)}) - \Psi^{(\pm)i\dagger}\Psi_i^{(\pm)} + \mathcal{F}^{(\pm)}(-\Psi^{(\pm)} + \mathcal{A}^{(\pm)})\right)\eta_\pm \rangle.
\end{aligned} \tag{C.11}$$

Using (A.18) and the dilaton field equation (A.16), we get

$$\begin{aligned}
\tilde{R} &= -\tilde{\nabla}^i(h_i) + \frac{1}{2}h^2 - 4(\tilde{\nabla}\Phi)^2 - 2h^i\tilde{\nabla}_i\Phi - \frac{3}{2}L^2 + \frac{5}{12}\tilde{H}^2 \\
&+ \frac{7}{2}e^{2\Phi}S^2 - \frac{5}{4}e^{2\Phi}\tilde{F}^2 + \frac{3}{4}e^{2\Phi}X^2 - \frac{1}{48}e^{2\Phi}\tilde{G}^2 - \frac{9}{2}e^{2\Phi}m^2.
\end{aligned} \tag{C.12}$$

One obtains, upon using the field equations and Bianchi identities,

$$\begin{aligned}
& \left( \frac{1}{4} \tilde{R} + \Gamma^i \tilde{\nabla}_i (-\Psi^{(\pm)} + \mathcal{A}^{(\pm)}) - \Psi^{(\pm)i} \Psi_i^{(\pm)} + \mathcal{F}^{(\pm)} (-\Psi^{(\pm)} + \mathcal{A}^{(\pm)}) \right) \eta_{\pm} \\
&= \left[ \left( \pm \frac{1}{4} \tilde{\nabla}_{\ell_1} (h_{\ell_2}) \mp \frac{1}{16} \tilde{H}^i_{\ell_1 \ell_2} L_i \right) \Gamma^{\ell_1 \ell_2} + \left( \pm \frac{1}{8} \tilde{\nabla}_{\ell_1} (e^{\Phi} X_{\ell_2 \ell_3}) + \frac{1}{24} \tilde{\nabla}^i (e^{\Phi} \tilde{G}_{i \ell_1 \ell_2 \ell_3}) \right. \right. \\
&\mp \frac{1}{96} e^{\Phi} h^i \tilde{G}_{i \ell_1 \ell_2 \ell_3} - \frac{1}{32} e^{\Phi} X_{\ell_1 \ell_2} h_{\ell_3} \mp \frac{1}{8} e^{\Phi} \tilde{\nabla}_{\ell_1} \Phi X_{\ell_2 \ell_3} - \frac{1}{24} e^{\Phi} \tilde{\nabla}^i \Phi \tilde{G}_{i \ell_1 \ell_2 \ell_3} \\
&\mp \frac{1}{32} e^{\Phi} \tilde{F}_{\ell_1 \ell_2} L_{\ell_3} \mp \frac{1}{96} e^{\Phi} S \tilde{H}_{\ell_1 \ell_2 \ell_3} - \frac{1}{32} e^{\Phi} \tilde{F}^i_{\ell_1} \tilde{H}_{i \ell_2 \ell_3} \left. \right) \Gamma^{\ell_1 \ell_2 \ell_3} \\
&+ \Gamma_{11} \left( \left( \mp \frac{1}{4} \tilde{\nabla}_{\ell} (e^{\Phi} S) - \frac{1}{4} \tilde{\nabla}^i (e^{\Phi} \tilde{F}_{i \ell}) + \frac{1}{16} e^{\Phi} S h_{\ell} \pm \frac{1}{16} e^{\Phi} h^i \tilde{F}_{i \ell} \pm \frac{1}{4} e^{\Phi} \tilde{\nabla}_{\ell} \Phi S \right. \right. \\
&+ \frac{1}{4} e^{\Phi} \tilde{\nabla}^i \Phi \tilde{F}_{i \ell} + \frac{1}{16} e^{\Phi} L^i X_{i \ell} \mp \frac{1}{32} e^{\Phi} \tilde{H}^{ij}_{\ell} X_{ij} - \frac{1}{96} e^{\Phi} \tilde{G}^{ijk}_{\ell} \tilde{H}_{ijk} \pm \frac{1}{16} e^{\Phi} m L_{\ell} \left. \right) \Gamma^{\ell} \\
&+ \left( \mp \frac{1}{4} \tilde{\nabla}_{\ell_1} (L_{\ell_2}) - \frac{1}{8} \tilde{\nabla}^i (\tilde{H}_{i \ell_1 \ell_2}) + \frac{1}{4} \tilde{\nabla}^i \Phi \tilde{H}_{i \ell_1 \ell_2} \pm \frac{1}{16} h^i \tilde{H}_{i \ell_1 \ell_2} \right) \Gamma^{\ell_1 \ell_2} \\
&+ \left( \pm \frac{1}{384} e^{\Phi} \tilde{G}_{\ell_1 \ell_2 \ell_3 \ell_4} L_{\ell_5} \pm \frac{1}{192} e^{\Phi} \tilde{H}_{\ell_1 \ell_2 \ell_3} X_{\ell_4 \ell_5} + \frac{1}{192} e^{\Phi} \tilde{G}^i_{\ell_1 \ell_2 \ell_3} \tilde{H}_{i \ell_4 \ell_5} \right) \Gamma^{\ell_1 \ell_2 \ell_3 \ell_4 \ell_5} \left. \right] \eta_{\pm} \\
&+ \frac{1}{2} (1 \mp 1) \left( h^i \tilde{\nabla}_i \Phi - \frac{1}{2} \tilde{\nabla}^i h_i \right) \eta_{\pm} . \tag{C.13}
\end{aligned}$$

Note that with the exception of the final line of the RHS of (C.13), all terms on the RHS of the above expression give no contribution to the second line of (C.11), using (C.10), since all these terms in (C.13) are anti-Hermitian and thus the bilinears vanish. Furthermore, the contribution to the Laplacian of  $\|\eta_+\|^2$  from the final line of (C.13) also vanishes; however the final line of (C.13) *does* give a contribution to the second line of (C.11) in the case of the Laplacian of  $\|\eta_-\|^2$ . We proceed to consider the Laplacians of  $\|\eta_{\pm}\|^2$  separately, as the analysis of the conditions imposed by the global properties of  $\mathcal{S}$  differs slightly in the two cases.

For the Laplacian of  $\|\eta_+\|^2$ , we obtain from (C.11):

$$\tilde{\nabla}^i \tilde{\nabla}_i \|\eta_+\|^2 - (2\tilde{\nabla}^i \Phi + h^i) \tilde{\nabla}_i \|\eta_+\|^2 = 2 \|\hat{\nabla}^{(+)} \eta_+\|^2 - (4\kappa + 16\kappa^2) \|\mathcal{A}^{(+)} \eta_+\|^2 . \tag{C.14}$$

This proves (2.17).

The Laplacian of  $\|\eta_-\|^2$  is calculated from (C.11), on taking account of the contribution to the second line of (C.11) from the final line of (C.13). One obtains

$$\tilde{\nabla}^i (e^{-2\Phi} V_i) = -2e^{-2\Phi} \|\hat{\nabla}^{(-)} \eta_-\|^2 + e^{-2\Phi} (4\kappa + 16\kappa^2) \|\mathcal{A}^{(-)} \eta_-\|^2 , \tag{C.15}$$

where

$$V = -d \|\eta_-\|^2 - \|\eta_-\|^2 h . \tag{C.16}$$

This proves (2.20) and completes the proof.

## Appendix D The geometry of $\mathcal{S}$

It is known that the vector fields associated with the 1-form Killing spinor bilinears given in (3.5) leave invariant all the fields of massive IIA supergravity. In particular for massive IIA horizons we have that  $\mathcal{L}_{K_a}g = 0$  and  $\mathcal{L}_{K_a}F = 0$ ,  $a = 1, 2, 3$ , where  $F$  denotes collectively all the fluxes of massive IIA supergravity, where  $K_a$  are given in (3.6). Solving these conditions by expanding in  $u, r$ , one finds that

$$\begin{aligned} \tilde{\nabla}_i \tilde{V}_j &= 0, \quad \tilde{\mathcal{L}}_{\tilde{V}} h = \tilde{\mathcal{L}}_{\tilde{V}} \Delta = 0, \quad \tilde{\mathcal{L}}_{\tilde{V}} \Phi = 0, \\ \tilde{\mathcal{L}}_{\tilde{V}} X &= \tilde{\mathcal{L}}_{\tilde{V}} \tilde{G} = \tilde{\mathcal{L}}_{\tilde{V}} L = \tilde{\mathcal{L}}_{\tilde{V}} \tilde{H} = \tilde{\mathcal{L}}_{\tilde{V}} S = \tilde{\mathcal{L}}_{\tilde{V}} \tilde{F} = 0. \end{aligned} \quad (\text{D.1})$$

Therefore  $V$  is an isometry of  $\mathcal{S}$  and leaves all the fluxes on  $\mathcal{S}$  invariant. Furthermore, one can establish the identities

$$\begin{aligned} -2 \|\eta_+\|^2 - h_i \tilde{V}^i + 2\langle \Gamma_+ \eta_-, \Theta_+ \eta_+ \rangle &= 0, \quad i_{\tilde{V}}(dh) + 2d\langle \Gamma_+ \eta_-, \Theta_+ \eta_+ \rangle = 0, \\ 2\langle \Gamma_+ \eta_-, \Theta_+ \eta_+ \rangle - \Delta \|\eta_-\|^2 &= 0, \quad \tilde{V} + \|\eta_-\|^2 h + d\|\eta_-\|^2 = 0, \end{aligned} \quad (\text{D.2})$$

which imply that  $\mathcal{L}_{\tilde{V}} \|\eta_-\|^2 = 0$ . These conditions are similar to those established for M-theory and IIA theory horizons in [7] and [8], respectively, but of course the dependence of the various tensors on the fields is different. In the special case that  $\tilde{V} = 0$ , the horizons are warped products of  $AdS_2$  with  $\mathcal{S}$ .

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